PROXIMITY STRUCTURES AND GRILL

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Abstract— The concept of proximity has already studied with meticulous care from different aspects and via various approaches. In this paper, we have continued the investigation of proximity in terms of a different concept via grill, where the deliberations in the article include certain characterizations. The associated discussions and results were carried out with grills as a prime supporting tool. The basic properties of the induced proximity have discussed here in some details. We have also shown various relations between the proximity induced via grill and the original proximity under some suitable conditions applied on the grill under consideration.

Index Terms— Grill, Ef-proximity space, Basic proximity
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1 Introduction

In principal, a proximity space is a non-empty set X with a binary relation δ between subsets with the intuitive meaning that AδB holds, when "A is near B" in some sense. The proximity relation satisfies axioms which are identical with some of the typical axioms of the connection relation. Each proximity space determines a natural topology with useful properties, and the theory possesses deep results, rich machinery and tools; the main work on proximity spaces is the book by Naimpally and Warrack [4].
The idea of grill was initiated by Choquet [1], thereafter it has been observed in connection with many mathematical investigations such as the theories of proximity spaces, compactifications etc,. Grills are extremely useful and convenient for many situations as a tool like filters. A grill G on a topological space X is defined as a collection of non-empty subsets of X such that:

(i) $A \in G$ and $A \subseteq B \subseteq X \rightarrow B \in G$.
(ii) $A, B \subseteq X$ and $A \cup B \in G \rightarrow A \in G$ or $B \in G$.

The current work aims to establish a generalized proximity using grill and Ef-proximity on a set. In section 2, all preliminaries and theorems of proximity structures and grills which will be needed in the sequel have been briefly mentioned. In section 3, $\varphi_G(\delta)$ and $\psi_G(\delta)$ operators on P(X) w.r.t. grill G and Ef-proximity δ, are defined and various properties of them are investigated. Also, the new generated proximity will be introduced. Furthermore, the relation between the new generated proximities and the new generated topologies using a given proximity or topology and grills will be discussed.

2. Preliminaries

This section is devoted to recall the known results concerning grill and proximity spaces. Many of the results below are translations of facts about grill and proximity spaces of which proofs are available in [1, 2, 4, 5]. Since space is at a premium, we will usually cite this source for those proofs which are known and not immediately obvious.

Proposition 2.1 Arbitrary union of grills is a grill, but the intersection of two grills is not generally a grill.

Definition 2.2 Let X be a set, then the principal grill generated by a non-empty subset A of X is $[A]=\{B\subseteq X : A\cap B\neq\emptyset\}$.

Definition 2.3 A basic proximity structure on a non-empty set X is a binary relation δ on P(X) such that:
(i) $A\delta B \rightarrow B\delta A$.
(ii) $(A\cup B)\,\delta C \leftrightarrow A\delta C$ or $B\delta C$.
(iii) $A\delta B \rightarrow A\neq\emptyset$ and $B\neq\emptyset$.
(iv) $A\cap B\neq\emptyset \rightarrow A\delta B$.

A basic proximity δ is said to be Ef-proximity, if it satisfies (where $\bar{\delta}$ means negation of δ):
$A\bar{\delta} B \rightarrow \exists E\subseteq X$ s.t. $A\bar{\delta} E$ and $(X-E)\,\bar{\delta} B$.

A^c represents the complement of a set A.

Proposition 2.4 Let (X,δ) be an Ef-proximity space and $A \subseteq X$, then:
(i) $Cl_\delta(A)=\{x\in X : U\cap A\neq\emptyset\ \forall\delta\text{-nbd.}U \text{ of } x\}=\{x\in X : x\delta A\}$ is a Kuratowski closure operator and the collection $\tau_\delta=\{A\subseteq X : Cl_\delta(A^c)=(A^c)\}$ is a topology on X which is a completely regular if, (X, δ) is an Ef-proximity space.
(ii) A is a δ-open set w.r.t. τ_δ, if $(x\bar{\delta}A^c\ \forall x\in A)$.
(iii) A is a δ-closed set w.r.t. τ_δ, if $(x\delta A \rightarrow x\in A)$.

Note that δ and τ_δ are called compatible.

Proposition 2.5 For every subset A and B of an Ef-proximity space (X,δ), then:
(iii) If AδB, $A\subseteq C$ and $B\subseteq D$, then CδD.

(iii) $Cl_\delta A \ \delta \ Cl_\delta B \leftrightarrow A\delta B$.

Definition 2.6 If δ_1 and δ_2 are two proximities on a set X, then δ_1 is called finer than δ_2 (in symbols $\delta_2<\delta_1$) if, $A\delta_1B$ implies $A\delta_2B$.

Theorem 2.7 (i) Let δ_1 and δ_2 be Ef-proximities on a set X, then $\delta_1 < \delta_2$ implies $\tau_{\delta 1}\subseteq\tau_{\delta 2}$.
(ii) Each completely regular topological structure τ on a set X generates a unique compatible proximity structure δ, given by $A\delta B$ iff $Cl\, A\cap Cl\, B\neq\emptyset \ \ \forall A, B\subseteq X$.
(iii) Let τ_1, τ_2 be two completely regular topologies on X, δ_1, δ_2 be the proximities generated by τ_1 and τ_2 respectively. Then, $\tau_1\subseteq \tau_2$ implies $\delta_1 < \delta_2$.

Definition 2.8 A subset B of any proximity space (X, δ) is called a δ-neighborhood of a set A (in symbols A<<B), if $A\bar{\delta}$ (X - B). The family $N(\delta, A)=\{B\subseteq X : A<<B\}$ is called the δ-nbd system of A.

Definition 2.9 Let (X, δ) be a proximity space and $x\in X$. Then, $G_\delta=\{U\subseteq X : U$ is δ-nbd. of $x\}$ is a grill on X generated by δ.

Throughout this paper, the space (X, δ, G) refers to an Ef-proximity space (X, δ) with a grill G on X.

3. Operator $\varphi_G(\delta)$ and Generated Proximity

Definition 3.1 Let (X, δ) be a proximity space and G be a grill on X. An operator φ_G: P(X) → P(X) (where P(X) stands for the power set of X), denoted by $\varphi_G(A, \delta)$ (for $A\in P(X)$) or $\varphi_G(A)$, called the operator associated with the grill G and the proximity δ, and is defined by $\varphi_G(A)=\{x\in X : U\cap A\in G \ \forall\delta\text{-nbd.}U \text{ of } x\}$.

Example 3.2 Let (X, δ, G) be an Ef-proximity space with a grill G and $A\subseteq X$. Then:
(i) If $G=P(X)\backslash\{\emptyset\}$, then $\varphi_G(A)=A$.
(ii) If G=[A], then $\varphi_{[A]}(A)=Cl_\delta A$.

Proposition 3.3 Let (X,δ, G) be an Ef-proximity space with a grill G and $A\subseteq X$ be δ-closed, then $\varphi_G(A)\subseteq A$.

Proof Let $A\subseteq X$ be a δ-closed set and $x\notin A$, then A^c is a δ-nbd. of x and $(A^c\cap A)=\emptyset\notin G$ .Consequently, $x\notin\varphi_G(A)$ and so $\varphi_G(A)\subseteq A$.

Theorem 3.4 Let (X, δ) be an Ef-proximity space and let G and H be two grills on X. For a subset A of X, the following statements are hold:
(i) $G\subseteq H \rightarrow \varphi_G(A)\subseteq \varphi_H(A)$.
(ii) $\varphi_{G\cup H}(A)= \varphi_G(A)\cup \varphi_H(A)$.

Proof (i) Obvious,
(ii) The inclusion $\varphi_G(A)\cup \varphi_H(A)\subseteq \varphi_{G\cup H}(A)$ follows directly from (i). Let $x\notin\varphi_G(A)\cup\varphi_H(A)$, then $x\notin\varphi_G(A)$ and $x\notin\varphi_H(A)$. Hence, $\exists U_1$, U_2 δ-nbds. of x s.t. $U_1\cap A\notin G$ and $U_2\cap A\notin H$. Let $U=U_1\cap U_2$ δ-nbd. of x s.t. $U\cap A\notin G$ and $U\cap A\notin H$. This follows that, $(U\cap A)\notin G\cup H$ and so $x\notin\varphi_{G\cup H}(A)$ i.e. $\varphi_{G\cup H}(A)\subseteq \varphi_G(A)\cup \varphi_H(A)$.

Next theorems discuss some of properties of the φ_G-operator.

Theorem 3.5 Let (X,δ, G) be an Ef-proximity space with a grill G. Then, for any A, B_X the following hold:
(i) $\varphi_G(A)\subseteq \varphi_G(B)$, if $A\subseteq B$.
(ii) $\varphi_G(A\cup B)=\varphi_G(A)\cup\varphi_G(B)$.
(iii) $\varphi_G(A\cap B)\subseteq\varphi_G(A)\cap\varphi_G(B)$.
(iv) $\varphi_G(A)\subseteq Cl_\delta(A)$.
(v) $\varphi_G(A)=\emptyset$, if $A\notin G$, moreover $\varphi_G(\emptyset)=\emptyset$.
(vi) $\varphi_G(A) - \varphi_G(B) \subseteq \varphi_G(A - B)$

Proof (i) Obvious from the definition of grill.
(ii) In view of (i) it suffices to show that $\varphi_G(A\cup B)\subseteq\varphi_G(A)\cup\varphi_G(B)$. Suppose $x\notin\varphi_G(A)\cup\varphi_G(B)$. Then, there are δ-nbds. U_1, U_2 of x such that $A\cap U_1 \notin G$ and $B\cap U_2 \notin G$. Hence, $(A\cap U_1)\cup(B\cap U_2)\notin G$. Now $U_1\cap U_2$ is a δ-nbd. of x and $(A\cup B)\cap(U_1\cap U_2) \subseteq(A\cap U_1)\cup(B\cap U_2)\notin G$. Consequently, $x\notin\varphi_G(A\cup B)$.
(iii) It is obvious in view of (i).
(iv) $x\notin Cl_\delta A$, then $x\bar{\delta}A$. Consequently, $\exists U=(X-A)$ δ-nbd.of x such that $U\cap A=\emptyset\notin G$ and so $x\notin\varphi_G(A)$. Thus $\varphi_G(A)\subseteq Cl_\delta A$.
(v) Obvious from the definition of the operator φ_G.
(vi) Let A, $B\subseteq X$ and $A=(A-B)\cup(A\cap B)$. Then, (ii) and (iii) implies that, $\varphi_G(A)\subseteq \varphi_G(A-B)\cup\varphi_G(B)$. Consequently, $\varphi_G(A) - \varphi_G(B) \subseteq\varphi_G(A-B)$.

Theorem 3.6 Let (X, δ, G) be an Ef-proximity space with a grill G. Then, $\forall A$, $B\subseteq X$ the following hold:
(i) $\varphi_G(A\cup B)=\varphi_G(A)= \varphi_G(A-B)$, if $B\notin G$.
(ii) $\varphi_G(A)=\varphi_G(B)$, if $(A-B)\cup(B-A)\notin G$.

Proof (i) Obvious by using Theorem 3.5.
(ii) Let $E=(A-B)\cup(B-A)\notin G$, then $A=(E-B)\cup(B-E)$. By using (ii) of Theorem 3.5. and (i) of this theorem, we have $\varphi_G(A)= \varphi_G(E - B)\cup\varphi_G(B - E)$ which implies $\varphi_G(A)= \varphi_G(E - B)\cup\varphi_G(B)=\varphi_G(B\cup E)= \varphi_G(B)$.

Theorem 3.7 Let (X,δ, G) be an Ef-proximity space with a grill G. Then, for any $A\subseteq X$ the following hold:
(i) $x\delta\varphi_G(A) \rightarrow x\delta A$.
(ii) $x\in\varphi_G(A) \leftrightarrow x\delta(A-E) \ \forall E\notin G$.

Proof (i) Let $x\delta\varphi_G(A)$, then by using Theorem 3.5, $x\delta Cl_\delta A$. Since δ is an Ef-proximity on X, then $x\delta A$.
(ii) Let $x\in\varphi_G(A)$, then $x\delta\varphi_G(A)$. By using of Theorem 3.6. $x\delta\varphi_G(A-E) \ \ \forall E\notin G$. Consequently, by using (i) of this Theorem, $x\delta(A-E) \ \forall E\notin G$. Conversely, assume $x\notin\varphi_G(A)$, then $\exists\delta$ -nbd. U of x s.t. $U\cap A\notin G$. Put $E=U\cap A$, then $(A-E)\subseteq(X-U)$ and so $x\bar{\delta}(A-E)$ for some $E\notin G$. Consequently, $x\in\varphi_G(A)$ iff $x\delta(A-E)\forall E\notin G$.

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Proposition 3.8 Let (X,δ, G) be an Ef-proximity space with a grill G and $A\subseteq X$. If U is a δ-nbd.of x, then, $U\cap\varphi_G(A)= U\cap\varphi_G(U\cap A)$.

Proof In view of (i) of Theorem 3.5. and $U\cap A\subseteq A$, then $\varphi_G(U\cap A)\subseteq\varphi_G(A)$ and so $U\cap\varphi_G(U\cap A)\subseteq U\cap\varphi_G(A)$. Let $x\notin\varphi_G(U\cap A)$ and U is a δ-nbd .of x, then ∃δ-nbd. V of x s.t. $(V\cap U\cap A)\notin G$. Since U is a δ-nbd. of x, then $W=V\cap U$ is a δ-nbd. of x and $(W\cap A)\notin G$. Consequently, $x\notin\varphi_G(A)$ and so $U\cap\varphi_G(A)\subseteq U\cap\varphi_G(U\cap A)$. Then, $U\cap\varphi_G(A)= U\cap\varphi_G(U\cap A)$.

Theorem 3.9 Let (X, δ, G) be a proximity space with a grill G and $A\subseteq X$, then $\varphi_G(\varphi_G(A))\subseteq\varphi_G(A)=Cl_\delta(\varphi_G(A))$.

Proof We shall show that $Cl_\delta(\varphi_G(A))\subseteq\varphi_G(A)$. Let $x\notin\varphi_G(A)$, then ∃U δ-nbd.of x such that $U\cap A\notin G$. By using (v) of Theorem 3.5., then $\varphi_G(U\cap A)=\emptyset$. By using Proposition 3.8., then $U\cap\varphi_G(A)=\emptyset$ and so $\varphi_G(A)\subseteq X\text{-}U$. Consequently, $x\bar{\delta}\ \varphi_G(A)$ which implies that $x\notin Cl_\delta(\varphi_G(A))$. Then, $Cl_\delta(\varphi_G(A))\subseteq\varphi_G(A)$ and $\varphi_G(A)=Cl_\delta(\varphi_G(A))$ i.e. $\varphi_G(A)$ is closed w.r.to δ. By using (iv) of Theorem 3.5. $\varphi_G(\varphi_G(A))\subseteq Cl_\delta(\varphi_G(A))$ and so $\varphi_G(\varphi_G(A))\subseteq\varphi_G(A)$.

Proposition 3.10 Let (X, δ, G_δ) be a proximity space with a grill G_δ and $x\in X$. Then:
(i) $\varphi_{G\delta}(X)=X$
(ii) $A\subseteq\varphi_{G\delta}(A)$ ∀δ-nbd. A of x.

Proof (i) Obvious
(ii) Let A be δ-nbd. of x, then $A\in G_\delta$. By using Proposition 3.8. and (i) of this proposition $A\cap\varphi_G(X)=A\cap\varphi_G(A\cap X)$ and hence $A=A\cap\varphi_G(A)$. Consequently, $A\subseteq\varphi_G(A)$.

Corollary 3.11. Let (X, δ, G_δ) be a proximity space with a grill G_δ, then for a non-empty subset A of X the following statements are hold:
(i) $(\varphi_G (A))^c\subseteq\varphi_G(A^c)$.
(ii) If A is δ-open, then $x\delta\varphi_G(A)$ iff $x\delta A$.

Proof Obvious in view of Theorems 3.5., 3.7. and Proposition 3.10.

Proposition 3.12 Let δ_1, δ_2 be two proximities on a non-empty set X and δ_2 is finer than δ_1. Then, for any grill G on X and any subset A of X, $\varphi_G(A,\delta_2)\subseteq\varphi_G(A, \delta_1)$.

Proof Let $A\subseteq X$ and $x\notin\varphi_G(A, \delta_1)$, then $\exists\delta_1$-nbd. U of x s.t. $U\cap A\notin G$. By hypothesis $\delta_1<\delta_2$, U is a δ_2-nbd. of x and $U\cap A\notin G$, and so $x\notin\varphi_G(A, \delta_2)$. Consequently, $\varphi_G(A, \delta_2)\subseteq\varphi_G(A, \delta_1)$.

Theorem 3.13 Let G be a grill on a space X. We define a map $\psi_G : P(X) \to P(X)$, by $\psi_G(A)=A\cup\varphi_G(\tau , A)$, (for $A\in P(X)$), is a Kuratowski closure operator. Moreover, $\psi_G (A)\subseteq Cl_\delta(A)$.

Proof From the definition of ψ_G, statements (ii) and (v) of Theorem 3.5. and Theorem 3.9.; $\psi_G(A)$ is a Kuratowski closure operator and $\psi_G (A)\subseteq Cl_\delta(A)$.

For a given Ef-Proximity space with grill (X, δ, G), a new proximity structure is generated by using the closure operator ψ_G as we see in the following theorem.

Theorem 3.14 Let A and B be subsets of a space (X, δ, G). Then, the relation δ^*_G generated by the closure operator ψ_G which is defined by $A\delta^*_G B$ iff $\psi_G (A)\cap \psi_G(B)\neq\emptyset$ is abasic proximity on X, which is finer then δ.

Proof δ* is a basic proximity on X follows directly from the definition of δ^*_G . Moreover, $A\delta^*_G B \leftrightarrow \psi_G (A) \cap \psi_G(B)\neq\emptyset \to Cl_\delta A\cap Cl_\delta B\neq\emptyset\to Cl_\delta A\delta Cl_\delta \mathbf{B}\leftrightarrow A\delta B$. Hence, $\delta<\delta^*_G$.

Theorem 3.15 Let (X, δ) be an Ef-proximity space and let G and H be grills on X with $G\subseteq H$, then $\delta^*_H<\delta^*_G$.

Proof Let $A, B\subseteq X$ and $A\delta^*_G B$, then $\psi_G (A)\cap \psi_G(B)\neq\emptyset$ Since $G\subseteq H$ it follows that $\psi_H (A) \cap \psi_H(B)\neq\emptyset$. Consequently, $A\delta^*_H B$ and so $\delta^*_H<\delta^*_G$.

The proof of the following theorems is obvious and omitted.

Theorem 3.16 Let (X, δ) be an Ef-proximity space with grills G and H on X. Then, $\delta^*_{H\cup G} <\delta^*_H\cup\delta^*_G$.

Theorem 3.17 Let δ* be a proximity relation on X generated by an Ef-proximity relation δ and a grill G on X, then:
(i) δ* is a discrete proximity, if $G=P(X) - \{\emptyset\}$.
(ii) If $G=[A]$, then $A\delta^*B \leftrightarrow Cl_\delta(A)\cap Cl_\delta(B)\neq\emptyset\quad \forall A, B\subseteq X$.

Some induced proximity and topological structures and some relations among these structures by using the operator φ_G.

Note that; for a topological space with grill (X, τ , G) the induced topology τ* [3], is the topology generated by the closure operator $\psi_G(A)=A\cup\varphi_G(\tau , A)$, where $\varphi_G(\tau , A)=\{x\in X : U\cap A\in G\ \forall\tau\text{-open nbd. U of } x\}$. Moreover, $\varphi_G(\tau , A)=\varphi_G(\tau^*, A)$.

The relation between δ_{τ^*} and $(\delta_\tau)^*$ have been investigated.

Lemma 3.18 Let (X, τ , G) be a completely regular topological space with grill G and δ_τ be a compatible proximity structure. Then, $\forall A\subseteq X$, $\varphi_G(\tau, A)=\varphi_G(\delta_\tau, A)$ and the equality holds if (X, τ) is a compact space.

Proof Let $x\notin\varphi_G(\delta_\tau, A)$, then there is δ_τ -nbd. U of x s.t. $U\cap A\notin G$. Since every δ_τ -nbd. is a τ -nbd. Then, there is τ-open nbd. V of x s.t. $V\subseteq U$ and $U\cap A\in G$. Hence, $x\notin\varphi_G (\tau, A)$ and so $\varphi_G(\tau , A)\subseteq\varphi_G(\delta_\tau , A)$. Let (X,τ) be a compact space and let $x\notin\varphi_G (\tau, A)$, then there isτ-open nbd. U of x s.t. $U\cap A\in G$. Since τ is a compact completely regular topology, then U is a δ-nbd.

of x. Consequently, $x\notin A^*(\delta_\tau , G)$ and so, $\varphi_G(\delta_\tau, A)\subseteq\varphi_G(\tau , A)$. This follows hat $\varphi_G(\tau, A)=\varphi_G(\delta_\tau, A)$.

Theorem 3.19 Let (X, τ, G) be a completely regular space with a grill, then the induced topology τ^* is compatible with a proximity coarser than $(\delta_\tau)^*$ and if the space (X,τ , G) is also compact, then τ^* is compatible with $(\delta_\tau)^*$.

Proof Obvious, by using Lemma 3.18.

Theorem 3.20 If (X,τ, G) is a compact completely regular space with a grill G and τ^* is a compact topology on X. Then, $\delta_{\tau^*}< (\delta_\tau)^*$.

Proof Since τ^* is compact, $\varphi_G(\delta_\tau, A)=\varphi_G(\tau, A)=\varphi_G (\tau^*,A)= \varphi_G (\delta_{\tau^*}, A)$. Consequently, $(\delta_\tau)^*= (\delta_{\tau^*})^*$. Also, since $\delta_{\tau^*}< (\delta_{\tau^*})^*$, then $\delta_{\tau^*}< (\delta_\tau)^*$.

Theorem 3.21 Let (X, δ, G) be an Ef-proximity space with a grill G and let $A\subseteq X$, then $\varphi_G(\tau_\delta, A)\subseteq\varphi_G(\delta, A)$.

Proof Let $x\notin\varphi_G (\delta, A)$, then there is a δ-nbd. U of x s.t. $U\cap A\notin G$, which implies there is a τ_δ-open nbd V of x s.t. $V\subseteq U$ and $U\cap A\notin G$. Hence, $x\notin\varphi_G (\tau_\delta, A)$.

Theorem 3.22 $\tau_{\delta^*} \subseteq (\tau_\delta)^*$.

Proof Let A be a τ_{δ^*}-closed, then $\forall x\notin A$, $x\delta^{*-}A$. i.e $\psi(\{x\})\cap\psi(A)=\emptyset$ $\forall x\notin A$ which implies that $x\notin\varphi_G (\delta, A)$. Consequently, by using Theorem 3.21. $x\notin\varphi_G(\tau_\delta, A)$ $\forall x\notin A$. This follows that $\varphi_G (\tau_\delta, A)\subseteq A$, then A is a $(\tau_\delta)^*$-closed.

References

[1] G. Choquet, Sur les notions de filtre et grille, Comptes Rendus Acad. Sci. Paris 224, 1947, 171-173.

[2] V.A. Efremovich, The geometry of proximity, Mat. sb. 31, 1952, 189-200.

[3] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Month. 97, 1990, 295-310.

[4] S.A. Naimpally and B. D.Warrack, Proximity spaces, Cambridge Tract. 1970.

[5] W. J. Thron, Proximity structure and grills, Math. Ann 206, 1973, 35-62.